

# [11-02-27A-T]

## Functions

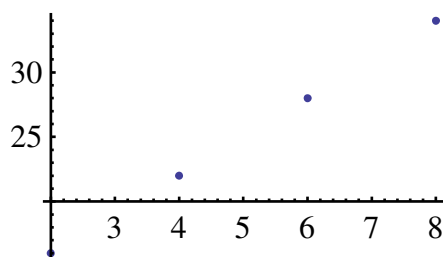
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**FUNCTION.** A *function* is a rule that pairs each element of a set called the *domain of the function* with exactly one element of another set called the *range of the function*.

**EXAMPLE 1.** What is the range of the function  $y = 2x + 1$ , domain  $\{-2, -1, 0, 1, 2\}$ ? Answer. The range is  $\{-3, -1, 1, 3, 5\}$ , because the rule  $y = 2x + 1$  pairs  $-2$  with  $-3$ ,  $-1$  with  $-1$ ,  $0$  with  $1$ ,  $1$  with  $3$ ,  $2$  with  $5$ .

Instead of writing all that, we will use some helpful notations. In print, you will often see the symbol  $\mathcal{D}$  for domain,  $\mathcal{R}$  for range. When we write by hand, we usually use Dmn for domain and Rng for range. Instead of writing “pairs  $-2$  with  $-3$ , pairs  $-1$  with  $-1$ ”, we write “ $-2 \rightarrow -3$ ,  $-1 \rightarrow -1$ ”.

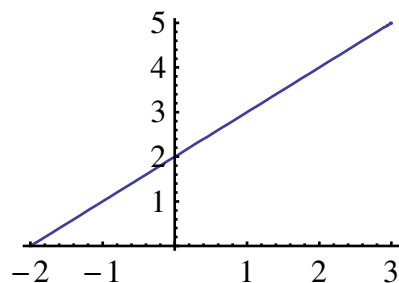
**EXAMPLE 2.** What is the range of the function  $y = 3x + 10$ ,  $\mathcal{D} = \{2, 4, 6, 8\}$ ? Answer.  $\mathcal{R} = \{16, 22, 28, 34\}$ , because  $2 \rightarrow 16$ ,  $4 \rightarrow 22$ ,  $6 \rightarrow 28$ ,  $8 \rightarrow 34$ . The graph of the function  $3x + 10$  is the following.



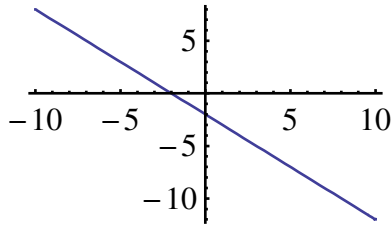
**EXAMPLE 3.** What is the range of the function  $y = x + 2$ ,  $\mathcal{D} = \{x : -2 \leq x \leq 3\}$ ? Answer.

$\mathcal{R} = \{y : 0 \leq y \leq 5\}$ . Certainly,  $-2 \rightarrow 0$ ,  $-1 \rightarrow 1$ . But what about when  $x = \frac{-1}{2}$ ? Of course,  $\frac{-1}{2} \rightarrow \frac{3}{2}$ .

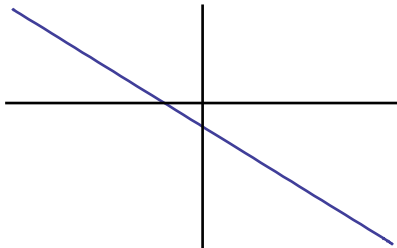
OK, then how about  $x = \frac{1}{4}$ ? It is obvious we will never succeed in listing all the pairings as in the previous examples, because there are infinitely many elements in the function's domain. A graph shows *all* the pairings.



EXAMPLE 4. What is the range of the function  $y = -x - 2$ ,  $\mathcal{D} = \{x: -10 \leq x \leq 10\}$ ?



So, when  $\mathcal{D} = \{x: -10 \leq x \leq 10\}$ ,  $\mathcal{R} = \{y: -12 \leq y \leq 8\}$ . Notice that the graph did not go up as far as  $y = 8$ , at least it is not obvious that it did. *But did that matter?* Suppose that for the graph you have only the following,



It is enough to know that the function takes its maximum value at  $x = -10$  and its minimum value at  $x = 10$  and that it takes all values in between. In fact, you could have known this merely by noting that since the slope of  $y = -x - 2$  is negative, the line runs down as  $x$  takes values from  $-10$  to  $10$ . Even so, you should form the habit of at least imagining the graph when you think about a function.

EXAMPLE 4. Recall that we have very nice notation for intervals of the  $x$ -axis. Thus, when  $y = 5x$  and  $\mathcal{D} = [-5, 5]$ ,  $\mathcal{R} = \{-25 \leq y \leq 25\}$ . It would be unusual to use interval notation for an interval of the  $y$ -axis, so we write  $\{-25 \leq y \leq 25\}$  not  $[-25, 25]$ .

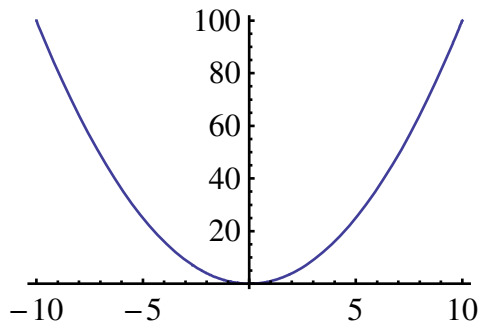
EXAMPLE 5. What is the range of  $y = 10x + 1$ ? Since no domain has been explicitly specified, we assume the largest domain for which the rule  $y = 10x + 1$  is defined. In this case, the domain is  $\mathbb{R}$ , all real numbers. The range is likewise  $\mathbb{R}$ , all real numbers.

EXAMPLE 6. Discuss the function  $y = -5x + 1$ ,  $\mathcal{D} = [-100, 50]$ . Answer. The domain of the function is  $\{-100 \leq x \leq 50\}$ , and its range is  $\mathcal{R} = \{y: -249 \leq y \leq 501\}$ . The function attains its maximum value of  $y = 501$  at  $x = -100$ , and its minimum value of  $y = -249$  at  $x = 50$ .

EXAMPLE 7. What are the maximum (minimum) values of  $y = 2x + 6$ ? There is no minimum and no maximum. The range is  $\mathbb{R}$ .

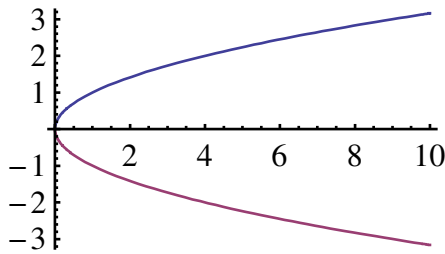
So far, we have looked at linear functions. It may be helpful to consider some other types of functions and a non-example of a function.

EXAMPLE 8. What is the range of  $y = x^2$ ? You will soon learn that the graph of  $y = x^2$  is as shown.



It certainly *appears* that the range of  $y = x^2$  on domain  $\mathcal{D} = \mathbb{R}$  is  $\{y : y \geq 0\}$ . In fact, this is true.

EXAMPLE 9. Discuss  $y^2 = x$ . The graph, as you will learn, is as shown.



Our conclusion is that  $y^2 = x$  is *not* even a function, because  $x = 4$  is paired with  $y = 2$  and  $y = -2$ . The requirement that a function pair each element of its domain with *exactly* one element of another set is violated. We can call  $y^2 = x$  a relation.

EXAMPLE 10. What is the domain of  $y = \frac{1}{x}$ . Answer. The domain of  $y = \frac{1}{x}$  is all real numbers *except* zero, because division by zero is undefined. We write this  $\mathcal{D} = \mathbb{R} - \{0\}$ , or  $\{x \in \mathbb{R} : x \neq 0\}$ , or, if it is understood we are considering real numbers, we might write only  $x \neq 0$ .

ORDERED PAIRS. Consider  $y = 2x + 1$ ,  $\mathcal{D} = \{-2, -1, 0, 1, 2\}$ . There is another way to write the fact that  $y = 2x + 1$  pairs  $-2$  with  $-3$ ,  $-1$  with  $-1$ ,  $0$  with  $1$ ,  $1$  with  $3$ ,  $2$  with  $5$ . We may write this:  $(-2, -3)$ ,  $(-1, -1)$ ,  $(0, 1)$ ,  $(1, 3)$ ,  $(2, 5)$ . Each of these is called an *ordered pair* because each shows the pairing of  $x$  with  $y$  in the order  $(x, y)$ . Of course, these are the same ordered pairs that give the address in the coordinate plane of the points whose coordinates make  $y = 2x + 1$ ,  $\mathcal{D} = \{-2, -1, 0, 1, 2\}$  true.

You will eventually learn that we can define a function as a set of ordered pairs such that no first element occurs more than once. For example, the function  $y = 2x + 1$ ,  $\mathcal{D} = \{-2, -1, 0, 1, 2\}$  can be written  $\{(-2, -3), (-1, -1), (0, 1), (1, 3), (2, 5)\}$ .

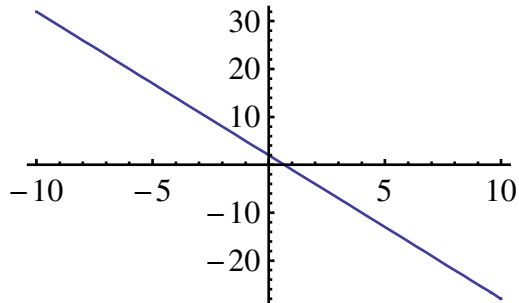
EXAMPLE 11. Is  $\{(0, 1), (1, 2), (3, 5), (4, 6), (10, 9)\}$  a function? If so, what is its domain and range? Answer. Yes. Domain is  $\{0, 1, 3, 4, 10\}$ . Range is  $\{1, 2, 5, 6, 9\}$ .

EXAMPLE 12. Is  $\{(0, 1), (1, 2), (3, 5), (3, 6), (4, 9)\}$  a function? If so, what is its domain and range? Answer. Not a function, because the 3 is paired with two numbers, 5, and 6.

FUNCTION NOTATION. Consider  $y = 2x + 1$ ,  $\mathcal{D} = \mathbb{R}$ . We emphasize that  $y$  is a function of  $x$  by writing  $y = f(x) = 2x + 1$ . Read “ $y$  equals  $f$  of  $x$  equals  $2x + 1$ ”. The *name* of the function is  $f$  and the *value* of the function at  $x$  is  $f(x)$ . When  $x = 3$ ,  $y = 7$ , so we write  $f(3) = 7$ . Read “ $f$  of 3 is 7”.

EXAMPLE 13. Let the function  $f$  be defined by  $f(x) = 10x - 4$ ,  $\mathcal{D} = \mathbb{R}$ . Evaluate the function  $f$  at  $x = -5, -1, 0, 2, 6$ . Answer.  $f(-5) = -54$ ,  $f(-1) = -14$ ,  $f(0) = 4$ ,  $f(2) = 16$ ,  $f(6) = 56$ .

EXAMPLE 14. Graph the function  $f$  defined by  $f(x) = -3x + 2$ ,  $\mathcal{D} = \mathbb{R}$ . Answer.



We understand the axes and the line continue in both directions without end.

EXAMPLE 15. Prove that the relation  $y = 2x + 3$ ,  $\mathcal{D} = \mathbb{R}$  is a function. To do this, we must show that every real number  $x$  is paired with one number and no more than one number  $y$ . Note that the statement “ $x$  is paired with no more than one number  $y$ ” is equivalent to the statement “If  $y_1 = y_2$  then  $x_1 = x_2$ , where  $x_1$  is paired with  $y_1$  and  $x_2$  is paired with  $y_2$ .”

*Proof.* Given that  $y = 2x + 3$ ,  $\mathcal{D} = \mathbb{R}$ , let  $x_1$  and  $x_2$  be any real numbers such that the relation pairs  $x_1$  with  $y_1$  and  $x_2$  with  $y_2$ . That is,  $y_1 = 2x_1 + 3$  and  $y_2 = 2x_2 + 3$ . Suppose that  $y_1 = y_2$ . Then,  $2x_1 + 3 = 2x_2 + 3$ . But this means that  $x_1 = x_2$ . Since every number in the domain of  $y = 2x + 3$  is paired with one and only one number,  $y = 2x + 3$ ,  $\mathcal{D} = \mathbb{R}$  is a function.

EXAMPLE 16. Prove that the sum of linear functions is associative.

*Proof.* Let  $f, g, h$  be three linear functions defined on  $\mathbb{R}$ . Then we may write  $f(x) = a_1x + b_1$ ,  $g(x) = a_2x + b_2$ ,  $h(x) = a_3x + b_3$ . We wish to show that  $(f + g) + h = f + (g + h)$ . Now,  $(f + g) + h = (a_1x + b_1 + a_2x + b_2) + a_3x + b_3$ . Since addition is associative, the parenthesis may be shifted. Thus,  $(a_1x + b_1 + a_2x + b_2) + a_3x + b_3 = a_1x + b_1 + (a_2x + b_2 + a_3x + b_3)$ . But this equals  $f + (g + h)$ .  $\square$

EXAMPLE 17. Show  $g(kx) = k \cdot g(x)$  when  $g$  is the linear function  $g(x) = mx$  and  $k$  is a constant not equal to zero.

*Proof.* Let  $g$  be the  $g(x) = mx$ . Now,  $g(kx) = m(kx) = k(mx) = k g(x)$ .  $\square$

[11-02-27B-T]

Functions

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*This problem set goes with the handout [11-02-27A-T].*

■ A.

- [1] Suppose  $f(x) = x + 5$ ,  $\mathcal{D} = \{-4, -3, -2, -1, 0, 1, 2\}$ . What is the range of  $f$ ?
- [2] Suppose  $g(x) = x + 5$ ,  $\mathcal{D} = \{x : x \in \mathbb{R}, x \geq 0\}$ . What is the range of  $f$ ?
- [3] Suppose  $f(x) = x + 5$ ,  $\mathcal{D} = \mathbb{Z}$ . (Recall that  $\mathbb{Z}$  is the set of integers.) What is the range of  $f$ ?
- [4] Suppose  $g(x) = 2x$ ,  $\mathcal{D} = \mathbb{Z}^+$ . (Recall that  $\mathbb{Z}^+$  is the set of positive integers.) What is the range of  $f$ ?
- [5] Suppose  $f(x) = 2x + 1$ ,  $\mathcal{D} = \mathbb{Z}^+$ . (Recall that  $\mathbb{Z}^+$  is the set of positive integers.) What is the range of  $f$ ?
- [6] Suppose  $h(x) = 2x - 3$ ,  $\mathcal{D} = \{x : x \in \mathbb{R}, -10 \leq x \leq 5\}$ . What is the range of  $f$ ?
- [7] Suppose  $f(x) = \frac{1}{2}x - 1$ ,  $\mathcal{D} = \{x : x \in \mathbb{R}, -1 \leq x \leq 1\}$ . What is the range of  $f$ ?
- [8] Suppose  $f(t) = -5t + 2$ ,  $\mathcal{D} = \{t : t \in \mathbb{R}, -10 \leq t \leq 10\}$ . What is the range of  $f$ ?
- [9] Suppose  $f(x) = -5x + 2$ ,  $\mathcal{D} = \{-2, 0, 3\}$ . What is the range of  $f$ ?
- [10] If the range of  $y = f(x) = 2x + 1$  is  $\{y \in \mathbb{R}, -12 \leq y \leq 4\}$ , what is the domain of  $f$ ?
- [11] What is the largest set that can be the domain of  $f(x) = \frac{2}{x-5}$ ?
- [12] If  $f(x) = 3x - 1$ , evaluate  $f(-2)$ ,  $f(0)$ ,  $f\left(\frac{1}{3}\right)$ ,  $f(2)$ ,  $f(a)$ .
- [13] If  $f(x) = 3x - 2$ , evaluate  $f(2a)$ .
- [14] Is  $\{(-5, 2), (-1, 8), (0, 6), (4, -12)\}$  a function?
- [15] Is  $\{(-5, 2), (-4, 2), (0, 2), (4, 2)\}$  a function?
- [16] Is  $\{(1, 2), (2, 3), (3, 4), (4, 5), (5, 6)\}$  a function?
- [17] Is  $\{(1, 2), (2, 1), (3, 4), (4, 3)\}$  a function?
- [18] Is  $\{(1, 2), (3, 4), (4, -6), (4, 7), (5, 6)\}$  a function?
- [19] Find the point at which the lines  $2x + 3y = 1$  and  $2x - 3y = 2$  intersect.

## ■ B.

For the following problems, it may be helpful to keep in mind the following two facts. [1] A function is a linear function of  $x$  if and only if it can be written in the form  $f(x) = ax + b$ , where  $a$  and  $b$  are constants. [2] The graph of a linear function is a straight line and a function whose graph is a straight line is a linear function.

[1] Prove that the sum of two linear functions is a linear function. (Hint: let one function be  $f(x) = ax + b$  and the other function be  $g(x) = cx + d$ .)

[2] Is the difference of two linear functions a linear function? If so, prove it. If not, provide a counterexample.

[3] Is the sum of two linear functions commutative? If so, prove it. If not, provide a counterexample.

[4] Prove that if  $f$  is the linear function  $f(x) = 3x$ , then  $f(2x) = 2f(x)$ .

[5] Prove that the product of a linear function and a constant is a linear function.

[6] Let functions  $f$  and  $g$  be defined by  $f(x) = x + 1$ ,  $\mathcal{D} = \mathbb{Z}^-$  and  $g(x) = 2x + 3$ ,  $\mathcal{D} = \mathbb{Z}^+$ . Explain why the sum of the functions  $f$  and  $g$  does not exist.

[7] Let functions  $f$  and  $g$  be defined by  $f(x) = x + 1$ ,  $\mathcal{D} = \{x \in \mathbb{Z}, x \geq 0\}$  and  $g(x) = 2x + 3$ ,  $\mathcal{D} = \{x \in \mathbb{Z}, x \leq 0\}$ . Find the function  $h$  which is the sum of the functions  $f$  and  $g$ , be sure to state the domain of  $h$ .

[8] Show that if  $f$  is a linear function, it is not always true that  $f(kx) = k \cdot f(x)$ , where  $k$  is a constant.

[9] Find the linear function  $f$  pairs 2 with 5 and 3 with 10.

[10] Prove that there can be no more than one linear function that pairs 2 with 5 and 3 with 10. (Hint: assume there are two such functions and show they must be identical.)

## ■ C.

[1] Many textbooks mention an informal test called *the vertical line test* to determine if a function is a linear function. According to the test, if every vertical line that intersects the graph of a function does so exactly once, then the function is a linear function. Explain why this is a reasonable test.

[2] A function  $y = f(x)$  is said to be a 1-1 function (pronounced “one to one function”) if and only if  $f$  is a function and no more than one value of  $x$  is paired with each value of  $y$ .

(a) Think of an informal test like the vertical line test mentioned in problem the preceeding problem that may be used to discover if a function is a 1-1 function. Illustrate your test by applying it to a linear function and to the function  $y = x^2$ .

(b) We have noted elsewhere that the statement “ $x$  is paired with no more than one number  $y$ ” is equivalent to the statement “If  $y_1 = y_2$  then  $x_1 = x_2$ , where  $x_1$  is paired with  $y_1$  and  $x_2$  is paired with  $y_2$ .” Find a similar statement that can be made of 1-1 functions.

(c) Use your statement in (b) to prove that all linear functions are 1-1 functions.